Spreading, vanishing and singularity for radially symmetric solutions of a Stefan-type free boundary problem (Developments of the theory of evolution equations as the applications to the analysis for nonlinear phenomena)

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Spreading, vanishing and singularity for radially symmetric solutions of a Stefan-type free boundary problem

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1 Introduction

We consider a free boundary problem for a reaction-diffusion equation:

$$\text{(FBP)} \begin{cases} u_t - d\Delta u = f(u), & t>0,\ g(t)<r<h(t), \\ u(t,g(t)) = 0,\ u(t,h(t)) = 0, & t>0, \\ g'(t) = -\mu u_r(t,g(t)), & t>0, \\ h'(t) = -\mu u_r(t,h(t)), & t>0, \\ g(0)=g_0,\ h(0)=h_0,\ u(0,r)=u_0(r), & g_0 \le r \le h_0, \end{cases}$$

where d, μ, g_0 and h_0 ($g_0 < h_0$) are positive constants, $r = |x|$ ($x \in \mathbb{R}^N$), $\Delta = \partial_r^2 + (N-1)\partial_r/r$ for $N \geq 2$, and the initial function u_0 satisfies

$$u_0 \in C^2(g_0,h_0) \cap C([g_0,h_0]), \quad u_0 > 0 \ \text{ in } \ (g_0,h_0), \quad u_0(g_0) = u_0(h_0) = 0.$$

Moreover the nonlinear function is assumed to satisfy

$$f \in C^1(\mathbb{R}), \quad f(0)=f(1)=0, \quad f(u)>0 \ \ (0<u<1), \quad f(u)<0 \ (u>1),$$
$$f'(0)>0, \quad f(u)/u \text{ is decreasing with respect to } u \in [0,1].$$

Problem (FBP) may be used to model the spreading of invasive or new species, where $u(t,r)$ represents the population density of the species that occupy a radially symmetric region denoted by

$$\Omega(t) = \{x \in \mathbb{R}^N;\ g(t) < |x| < h(t)\}.$$

The free boundaries $r = g(t), h(t)$ imply the spreading front of the species, whose behaviors are determined by Stefan conditions $g'(t) = -\mu u_r(t,g(t))$, $h'(t) = -\mu u_r(t,h(t))$, respectively. It will be shown that $g(t)$ is decreasing and $h(t)$ is increasing with respect to $t>0$, and hence $\Omega(t)$ is expanding in $t>0$.

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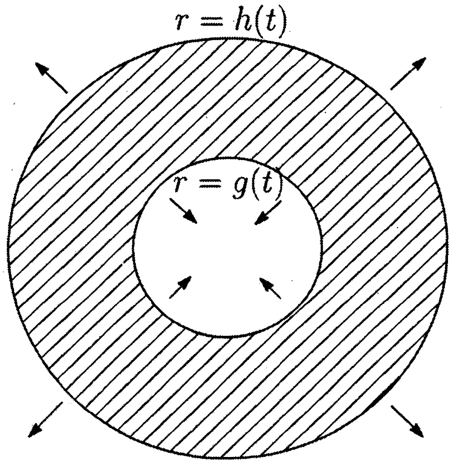


Figure 1. $\Omega(t)$ and free boundaries ($N = 2$)

This kind of free boundary problem was first proposed by Du-Lin [3] for $N = 1$:

$$\begin{cases} u_t - du_{xx} = u(a - bu), & t > 0,\ 0 < x < h(t), \\ u_x(t,0) = 0,\ u(t,h(t)) = 0, & t > 0, \\ h'(t) = -\mu u_x(t,h(t)), & t > 0, \\ h(0) = h_0,\ u(0,x) = u_0(x), & 0 \leq x \leq h_0, \end{cases} \tag{1.1}$$

where a and b are positive constants, d, μ and h_0 are defined as in (FBP), and u_0 satisfies $u_0 \in C^2(0,h_0) \cap C([0,h_0])$, $u_0 > 0$ in $(0,h_0)$, $u_0'(0) = u_0(h_0) = 0$. They proved the global existence and uniqueness of solutions to (1.1), and showed the spreading-vanishing dichotomy for asymptotic behaviors of solutions. It means that, for any solution (u,h) of (1.1), either (i) or (ii) occurs as t tends to infinity:

(i) Spreading: $\lim_{t\to\infty} h(t) = \infty$, $\lim_{t\to\infty} u(t,x) = a/b$ locally uniformly in $[0,\infty)$;

(ii) Vanishing: $\lim_{t\to\infty} h(t) \leq (\pi/2)\sqrt{d/a}$, $\lim_{t\to\infty} \|u(t,\cdot)\|_{C(0,h(t))} = 0$.

Here spreading implies the species succeed to establish themselves, while vanishing implies the extinction of the species. The number $(\pi/2)\sqrt{d/a}$ is called a threshold number in the sense that, once the free boundary reaches this number, spreading necessarily occurs. They also showed that, when spreading occurs, $h(t)/t$ converges to a constant as $t \to \infty$. This result implies that the spreading speed becomes almost constant in sufficiently large time.

After the work of Du-Lin [3], the free boundary problem has been studied by many researchers. Kaneko-Yamada [13] replaced Neumann boundary condition $u_x(t,0) = 0$ in (1.1) with Dirichlet boundary condition $u(t,0) = 0$ and gave sufficient conditions for spreading and vanishing. They also considered a bistable problem, where the nonlinear function of the problem is replaced by $u(u-c)(1-u)$ for $0 < c < 1/2$, and showed that such a threshold does not appear in the bistable problem. There are a lot of papers on one-dimensional free boundary problem (cf. Du-Lou [4], Du-Matsuzawa-Zhou [6], Gu-Lin-Lou [8], Guo-Wu [9], Kaneko-Oeda-Yamada [11], Kaneko-Matsuzawa [12], Liu-Lou [15], Wang [16] etc.). However there are only a few papers which deal with multi-dimensional free boundary problem (cf. Du-Guo [1], Du-Guo [2], Du-Matano-Wang [5], Kaneko [10]).

The situation is completely different in multi-dimensional free boundary problems. When $N \geq 2$, the geometric profile of free boundary relates strongly to the regularity of solutions. For example, when some parts of the free boundary connect each other, singularity appears for the density function. Then we can not deal with classical solution afterwards. However, introducing a weak form, we can consider the problem for all time. In (FBP), such a phenomenon actually occurs according to initial data and parameters in the equations.

The purpose of this paper is to introduce some results of Kaneko-Yamada [14] where the following contents are discussed:

(i) Existence and uniqueness of classical/weak solutions for (FBP);

(ii) Generation of singularity and regularity of weak solutions;

(iii) Spreading and vanishing in multi-dimensional problem (FBP).

Let (u, g, h) be solutions of (FBP) and $\Omega(t) = \{x \in \mathbb{R}^N;\ g(t) < |x| < h(t)\}$. Throughout this paper, we employ the notion of spreading, vanishing and singularity in the following sense:

(i) *Spreading* is the case where $\cup_{t>0}\Omega(t) = \mathbb{R}^N$ and $\lim_{t\to\infty} u(t,r) = 1$ uniformly in any compact set of $[0,\infty)$;

(ii) *Vanishing* is the case where $\cup_{t>0}\Omega(t)$ is a bounded set in $\mathbb{R}^N$ and $\lim_{t\to\infty} \|u(t,\cdot)\|_{C(g(t),h(t))} = 0$;

(iii) *Singularity* is the case where there exists a number $T^* \in (0,\infty]$ such that $\lim_{t\to T^*} g(t) = 0$.

We obtain the existence and uniqueness of classical solutions to (FBP) until inner boundary $g(t)$ reaches the origin and singularity appears. We continue to consider the problem afterwards by introducing a weak formulation, and moreover the weak solutions recovers smoothness immediately after singularity appears. Hence we study spreading and vanishing for classical solutions. Furthermore it will be shown that, if $\lim_{t\to\infty} h(t) = \infty$, then singularity appears at a finite time. We can refer details of proofs to [14].

The paper is organized as follows: in Section 2 we give main results for (FBP). This section is divided into two subsections; the former one relates to the existence and uniqueness of solutions to (FBP) and the latter is concerned with asymptotic behaviors of solutions.

2 Main Results

2.1 Existence and uniqueness of solutions

In this section we show the existence and uniqueness of solutions to (FBP). The assertions are summarized as follows:

- Let $T \in (0, \infty]$ satisfy $\lim_{t \to T} g(t) > 0$. Then there exists a unique local classical solution for $0 < t < T$, $g(t) < r < h(t)$. In other words u, u_r, u_{rr} and u_t are continuous for $0 < t < T$, $g(t) < r < h(t)$. Moreover the classical solution is extended to some time T^* when $g(t)$ reaches the origin.

- There exists a unique weak solution in the sense of Definition 1 for all time. This fact implies that we can solve the free boundary problem after $g(t)$ arrives at the origin at $t = T^*$, and that a weak solution is identical with a classical solution for $0 < t < T^*$.

- Every weak solution recovers smoothness for $T > T^*$. That means u, u_r, u_{rr} and u_t are continuous for $t > T^*$, $g(t) < r < h(t)$.

We have the local existence of a unique classical solution to (FBP).

Theorem 1. *For any given $\alpha \in (0,1)$, there exists a number $T > 0$ depending on g_0, h_0, α and $\|u_0\|_{C^2(g_0,h_0)}$ such that* (FBP) *has a unique solution (u, g, h) satisfying*

$$(u, g, h) \in \{C^{\frac{(1+\alpha)}{2}, 1+\alpha}(\overline{\Omega}_T) \cap C^{1+\frac{\alpha}{2}, 2+\alpha}(\Omega_T)\} \times C^{1+\frac{\alpha}{2}}[0, T] \times C^{1+\frac{\alpha}{2}}[0, T],$$

where $\Omega_T = \{(t, r) \in \mathbb{R}^2 \mid 0 < t \leq T,\ g(t) < r < h(t)\}$.

In the following theorem, we give the boundedness of solutions and monotonicity of the free boundaries, and show the time interval such that the classical solution exists.

Theorem 2. *Let T be any positive constant such that $g(T) > 0$. Then it holds that*

$$0 < u(t, r) \leq C_1 \ \text{in}\ \Omega_T \quad \text{and} \quad -\infty < g'(t) < 0 < h'(t) \leq \mu C_2, \ \text{for}\ 0 < t \leq T,$$

where constants C_1 and C_2 are independent of T, and Ω_T is the same as that of Theorem 1. Moreover the classical solution exists for $t \in (0, T_{max})$, where T_{max} is a positive constant that satisfies $T_{max} = \infty$ and $\lim_{t \to T_{max}} g(t) > 0$, or $T_{max} \in (0, \infty]$ and $\lim_{t \to T_{max}} g(t) = 0$.

We will introduce weak solutions to (FBP), referring to Du-Guo [2] and Friedman [7].

Definition 1. Let $G_T = (0,T) \times G$ for some $T > 0$ and bounded domain G satisfying $[0, h_0] \subset G \subset [0, \infty)$. A function $u(t,r)$ is called *a weak solution* to (FBP) over G_T when it satisfies

- $u \in H^1(G_T) \cap L^\infty(G_T), \quad u \geq 0 \ \text{ in } \ G_T,$
- $$\iint_{G_T} d(r^{N-1}u_r\phi_r) - r^{N-1}\alpha(u)\phi_t \, drdt - \int_G r^{N-1}\alpha(\tilde{u}_0)\phi_0 \, dr = \iint_{G_T} r^{N-1} f(u)\phi \, drdt \tag{2.1}$$

for any $\phi \in C^1(G_T)$ satisfying $\phi = 0$ for $(\{T\} \times G) \cup ([0,T] \times \partial G)$ and $\phi_0(r) := \phi(0,r)$. In (2.1), α and $\tilde{u}_0$ are given by

$$\alpha(u) = \begin{cases} u, & u > 0, \\ u - d/\mu, & u \leq 0, \end{cases} \qquad \tilde{u}_0 = \begin{cases} u_0, & r \in [g_0, h_0], \\ 0 & r \in G \setminus [g_0, h_0]. \end{cases}$$

We can apply a result in [2] to (2.1) to obtain the following result on the global existence of unique weak solutions.

Proposition 1. *For any $T > 0$, let $G \supset [0, h_0]$ be a sufficiently large domain. Then there exists a unique weak solution for* (FBP) *over $[0,T] \times G$.*

Remark 1. *By a comparison principle for the free boundary problem, one can choose a suitably large domain G such that G includes $[0, h(T)]$.*

We provide a relation between classical solutions and weak solutions.

Proposition 2. *The following results hold true:*

(i) *Let $u = u(t,r)$ be a classical solution to* (FBP). *Then a function*

$$v(t,r) = \begin{cases} u(t,r), & (t,r) \in \cup_{0<t<T}\{t\} \times (g(t), h(t)), \\ 0, & (t,r) \in \cup_{0<t<T}\{t\} \times (G \setminus (g(t), h(t))) \end{cases}$$

is a weak solution to (FBP) *over $G_T = (0,T) \times G$.*

(ii) *Let v be a weak solution to* (FBP) *over $G_T = (0,T) \times G$, and let $h, g \in C^1(0,T)$ $(g(t) < h(t)$ for $0 \leq t \leq T)$ satisfy*

$$\{r \in G, \ g(t) < r < h(t)\} = \{r \in G, \ v(t,r) > 0\},$$
$$\{r \in G, \ r \leq g(t), h(t) \leq r\} = \{r \in G, \ v(t,r) = 0\}$$

for $0 \leq t \leq T$. If a function u satisfies the following properties,

- *$u = v$ for $(t,r) \in \cup_{0<t<T}\{t\} \times [g(t), h(t)]$,*
- *u, u_r is continuous for $(t,r) \in \cup_{0\leq t<T}\{t\} \times [g(t), h(t)]$,*
- *u_{rr}, u_t is continuous for $(t,r) \in \cup_{0<t<T}\{t\} \times (g(t), h(t))$,*

then (u, g, h) *is a classical solution to* (FBP).

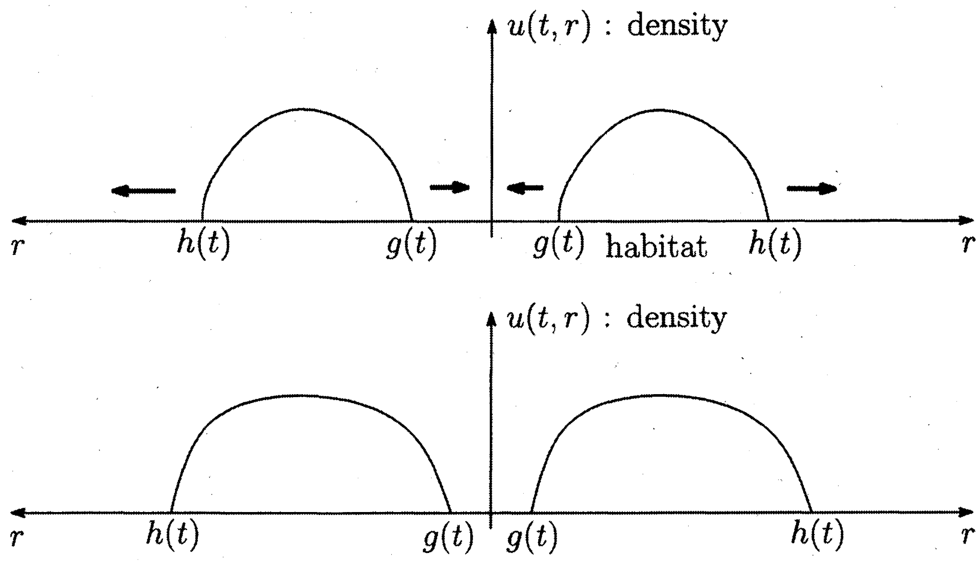
The following theorem assures any weak solution must become smooth immediately after singularity appears.

Theorem 3. *Assume that there exists a constant* $T^* > 0$ *satisfying* $\lim_{t\to T^*} g(t) = 0$. *Then any weak solution must be in* $C^{1,2}(D_{T^*})$, *where* $D_{T^*} = \cup_{t>T^*}\{t\} \times (0, h(t))$.

2.2 Spreading, vanishing and singularity

In this section we study the asymptotic behaviors of solutions to (FBP). With the help of Theorem 3, we may consider the classical solutions in large time, which makes it easier to investigate spreading and vanishing. The main results of this section are summarized as follows:

- If the outer boundary expands to infinity (i.e. $\lim_{t\to\infty} h(t) = \infty$), then the inner boundary reaches the origin at a finite time (i.e. $\lim_{t\to T^*} g(t) = 0$ for $T^* < \infty$).
- Spreading-vanishing dichotomy holds true for (FBP) in the sense of Theorem 5.
- There are some sufficient conditions for spreading and vanishing. If initial habitat is larger than the threshold value, or population density is sufficiently large, then spreading occurs. On the other hand, if initial habitat and the density are small, then vanishing occurs.



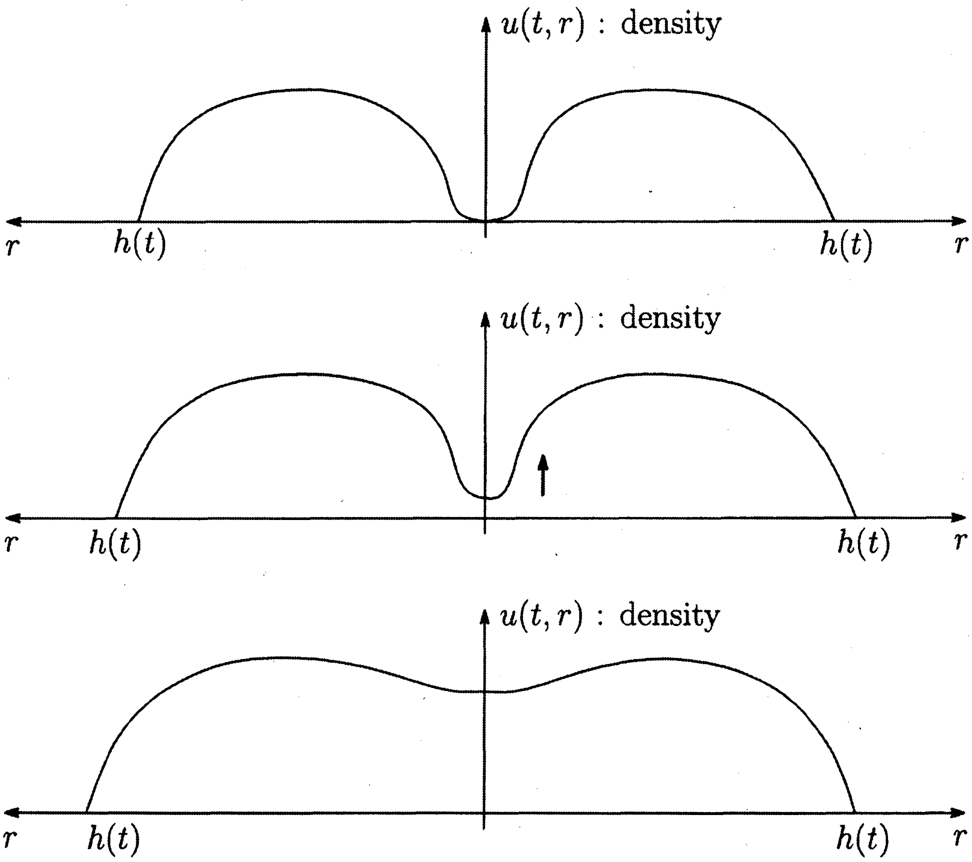


Figure 2. A profile of solution that generates singularity

The following theorem shows the generation of singularity.

Theorem 4. *If the solution* (u,g,h) *satisfies* $\lim_{t\to\infty} h(t) = \infty$, *then there exists a finite value* $T^* \in (0,\infty)$ *such that* $\lim_{t\to T^*} g(t) = 0$.

We will prepare some threshold numbers that play important roles. Let Ω be a bounded domain in $\mathbb{R}^N$. Denote by $\lambda_1 = \lambda_1(d;\Omega)$ the least eigenvalue for

$$\begin{cases} -d\Delta\phi = \lambda\phi, & x \in \Omega, \\ \phi = 0, & x \in \partial\Omega. \end{cases}$$

It is well known that $\lambda_1(d;\Omega)$ is continuous with respect to d and Ω, and $\lambda_1(d;\Omega_1) > \lambda_1(d;\Omega_2)$ if $\Omega_1 \subset \Omega_2$ $(\Omega_1 \neq \Omega_2)$. Let Ω be a ball with radius $l > 0$, that is, $\Omega = B_l := \{x \in \mathbb{R}^N;\ |x| < l\}$. Then $\lambda_1(d;B_l)$ is decreasing with respect to l and satisfies

$$\lim_{l\to 0+} \lambda_1(d;B_l) = +\infty, \quad \lim_{l\to+\infty} \lambda_1(d;B_l) = 0.$$

Hence there exists a unique number R_0^* such that

$$f'(0) = \lambda_1(d;B_{R_0^*}), \ f'(0) > \lambda_1(d;B_l) \ \text{ for } \ l > R_0^*.$$

We now replace Ω to $B_l \setminus B_{g(t)}$. Similarly we find $B_{l_1} \setminus B_{g(t_1)} \subset B_{l_2} \setminus B_{g(t_2)}$ for $t_1 \leq t_2$, $l_1 \leq l_2$ (because $g(t)$ is decreasing) and determine a unique positive number $R^* = R^*(d, g(t))$ for each $t \geq 0$ which satisfies

$$f'(0) = \lambda_1(d; B_{R^*} \setminus B_{g(t)}),\ \ f'(0) > \lambda_1(d; B_l \setminus B_{g(t)}) \ \text{ for } \ l > R^*.$$

The following proposition shows the dependence of $R^*(d, g(t))$ on d and t.

Proposition 3. *The following results hold for $R^*(d, g(t))$.*

(i) *$R^*(d, g(t))$ is monotone decreasing with respect to $t > 0$ and monotone increasing with respect to $d > 0$.*

(ii) *$R^*(d, g(t))$ is continuous for d and t. Moreover if there exists a number $T^* > 0$ such that $\lim_{t \to T^*} g(t) = 0$, then $\lim_{t \to T^*} R^*(d, g(t)) = R_0^*$.*

The following theorem provides spreading, vanishing and singularity for the free boundary problem.

Theorem 5. *Let (u, g, h) be any solution to* (FBP). *Then either* (i) *or* (ii) *holds true:*

(i) Spreading: $\cup_{t>0} \overline{\Omega}(t) = \mathbb{R}^N$, $\lim_{t \to \infty} u(t, r) = 1$ *uniformly in any bounded subset of* $[0, \infty)$;
Singularity: there exists finite value $T^* < \infty$ such that $\lim_{t \to T^*} g(t) = 0$;

(ii) Vanishing: *If $g_\infty := \lim_{t \to \infty} g(t) > 0$ (resp. $g(T_1) = 0$ for some $T_1 < \infty$), then $\cup_{t>0} \overline{\Omega}(t) \subset \overline{B}_{R^*_\infty} \setminus B_{g_\infty}$ $\lim_{t \to \infty} \|u(t, \cdot)\|_{C(g(t),h(t))} = 0$ (resp. $\cup_{t>0} \overline{\Omega}(t) \subset \overline{B}_{R^*_\infty}$, $\lim_{t \to \infty} \|u(t, \cdot)\|_{C(g(t),h(t))} = 0$), where $R^*_\infty := \lim_{t \to \infty} R^*(d, g(t))$. Moreover, for some $\beta > 0$, $\|u(t, \cdot)\|_{C(g(t),h(t))} = O(e^{-\beta t})$ as $t \to \infty$.*

We provide a sufficient condition for singularity.

Proposition 4. *If $h_0 \geq R^*(d, g_0)$, then singularity appears at a finite time, and spreading occurs as $t \to \infty$.*

The following theorem gives sufficient conditions for spreading and vanishing concerning on initial data.

Theorem 6. *Assume $h_0 < R^*(d, g_0)$. Let a smooth function $\phi = \phi(r)$ satisfy $\phi(g_0) = \phi(h_0) = 0$. Then there exists a positive number $\sigma^* \in [0, \infty]$ such that*

- *If $u_0 > \sigma^* \phi$, then singularity appears and spreading occurs;*
- *If $u_0 \leq \sigma^* \phi$, then vanishing occurs.*

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References

[1] Y. Du and Z. M. Guo, Spreading-vanishing dichotomy in the diffusive logistic model with a free boundary, II, J. Differential Equations, **250** (2011), pp. 4336-4366.

[2] Y. Du and Z. M. Guo, The Stefan problem for the Fisher-KPP equation, J. Differential Equations, **253** (2012), pp. 996-1035.

[3] Y. Du and Z. G. Lin, Spreading-vanishing dichotomy in the diffusive logistic model with a free boundary, SIAM J. Math. Anal., **42** (2010), pp. 377-405.

[4] Y. Du and B. Lou, Spreading and vanishing in nonlinear diffusion problems with free boundaries, J. Eur. Math. Soc., **17** (2015), pp. 2673-2724.

[5] Y. Du, H. Matano and K. Wang, Regularity and asymptotic behavior of nonlinear Stefan problems, Arch. Rational. Mech. Anal., **212** (2014), pp. 957-1010.

[6] Y. Du, H. Matsuzawa and M. Zhou, Sharp estimate of the spreading speed determined by nonlinear free boundary problems, SIAM J. Math. Anal., **46** (2014), pp. 375-396.

[7] A. Friedman, The Stefan problem in several space variables, Trans. Amer. Math. Soc. **132** (1968), pp. 51-87.

[8] H. Gu, Z. G. Lin and B. Lou, Different asymptotic spreading speeds induced by advection in a diffusion problem with free boundaries, Proc. Amer. Math. Soc., **143** (2015), pp. 1109-1117.

[9] J. S. Guo and C. H. Wu, On a free boundary problem for a two-species weak competition system, J. Dyn. Diff. Equat., **22** (2012), pp. 873-895.

[10] Y. Kaneko, Spreading and vanishing behaviors for radially symmetric solutions of free boundary problems for reaction-diffusion equations, Nonlinear Analysis: Real World Applications, **18** (2014), pp. 121-140.

[11] Y. Kaneko, K. Oeda and Y. Yamada, Remarks on spreading and vanishing for free boundary problems of some reaction-diffusion equations, Funkcial. Ekvac., **57** (2014), pp. 449-465.

[12] Y. Kaneko and H. Matsuzawa, Spreading speed and sharp asymptotic profiles of solutions in free boundary problems for nonlinear advection-diffusion equations, J. Math. Anal. Appl., **428** (2015), pp. 43-76.

[13] Y. Kaneko and Y. Yamada, A free boundary problem for a reaction-diffusion equation appearing in ecology, Adv. Math. Sci. Appl., **21** (2011), pp. 467-492.

[14] Y. Kaneko and Y. Yamada, Generation of singularity for a Stefan-type free boundary problem of a reaction-diffusion equation with non-convex initial domain, preprint.

[15] X. Liu and B. Lou, Asymptotic behavior of solutions to diffusion problems with robin and free boundary conditions , Math. Model. Nat. Phenom., **8** (2013), pp. 18-32.

[16] M. Wang, On some free boundary problems of the prey-predator model, J. Differential Equations, **256** (2014), pp. 3365-3394.